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# On the Length Biased Quasi-Transmuted Uniform Distribution

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**Abstract**

This study considered another modified Uniform Distribution, the Length Biased Quasi-Transmuted Uniform distribution. Some essential properties of the modified distribution were derived to show case its flexibility and tractability. The result indicates that the Length Biased Quasi-Transmuted Uniform distribution is a competent addition to the existing pool of generalized distributions.

***Keywords:*** *Length biased quasi-transmuted uniform distribution, Moment generating function, Cumulant generating function, Hazard function, Order statistics*.

## 1.0 Introduction

Development of generalized families of distributions is as old as Statistics itself. This has always been a growing research focus of statisticians, applied mathematicians, physicists and engineers. The core aim in this task is to reveal the intrinsic flexibilities inherent in the extended cum generalized distributing. Explorations of the properties of these distributions after their development from selected baseline distributions are equally of significant interest.

Existing distributions may fit badly the dataset under consideration and recourse must be made to generalized distributions [1]. Several of these generalized distributions exist in literature. Examples include New Extended Burr Type X distribution by [2]; Extended New Generalized Exponential distribution by [3]; Exponentiated Generalized Extended Exponential distribution by [4]; Alpha Power Transformed Generalized Exponential distribution by [5]; Extended Weighted Exponential distribution by [6]; Type I Generalized Exponential distribution by [7]; Marshall-Olkin Burr Type XII distribution by [8]; Scaled Burr Type X distribution by [9]; Burr Type X distribution by [10]; Extended Weibull Compound distribution by [11]; Extended Exponential distribution by [12]; Extended Uniform distribution by [13]; Negative Binomial Marshall-Olkin Rayleigh distribution by [14]; Extended Burr Type III distribution by [15]; Quasi-Transmuted Uniform distribution by [16] and Quasi-Transmuted Complementary Mukherjee-Islam distribution by [17].

This study however considers the Length Biased Quasi-Transmuted Uniform Distribution (LBQTUD) following the previous contributions to knowledge of the following authors: ([18]; [19]; [20]; [21]; [21] and [22]).

## 2.0 The Length Biased Quasi-Transmuted Uniform Distribution

The probability density function (pdf) of the Quasi-Transmuted distribution according to Osowole and Ayoola (2019) is defined as

$$g^*(x) = 2ax + (1-a), 0 < x < 1; a \in (0, \frac{1}{2}, 1) \qquad \text{.......................................(1.0)}$$

with cumulative distribution function (cdf) defined as

$$G^*(x) = ax^2 + (1-a)x, 0 < x < 1; a \in (0, \frac{1}{2}, 1) \qquad \text{.......................................(2.0)}$$

A length biased distribution according to Subramanian and Rather (2018) is a weighted distribution $f_w(x)$ defined as

$$f_w(x) = \frac{w(x) f(x)}{E[w(x)]}, x > 0 \qquad \text{.......................................(3.0)}$$

Where w(x) is non-negative such that $E[w(x)] = \int_{-\infty}^{+\infty} w(x) f(x) dx < \infty$

We note that (3.0) becomes a length biased distribution when w(x) is chosen such that w(x) = x and it becomes a size biased distribution when w(x) is chosen such that $w(x) = x^c$. Combining (1.0) and (3.0), the probability density function of the length biased quasi transmuted uniform distribution is given as

$$g_{LBQTUD}(x) = \frac{x[2ax + (1-a)]}{E[x]}, 0 < x < 1; a \in (0, \frac{1}{2}, 1)$$

$$= \frac{2ax^2 + (1-a)x}{[(a+3)/6]}$$

$$= \frac{12ax^2 + 6(1-a)x}{[(a+3)]}, 0 < x < 1; a \in (0, \frac{1}{2}, 1) \qquad \text{.......................................(4.0)}$$

(4.0) shall henceforth be referred to in this study as the Length Biased Quasi-Transmuted Uniform Distribution (LBQTUD). The cumulative distribution function of (4.0) is defined as

$$G_{LBQTUD}(x) = \int_0^x g_{LBQTUD}(t) dt$$

$$= \int_0^x \frac{12at^2 + 6(1-a)t}{a+3} dt$$

$$= \frac{4ax^3 + 3(1-a)x^2}{a+3}, 0 < x < 1; a \in (0, \frac{1}{2}, 1) \qquad \text{.......................................(5.0)}$$

## 2.1 Some Characterizations of the Length Biased Quasi-Transmuted Uniform Distribution

### 2.1.1 Moments

The $k^{th}$ raw moment of a random variable X from the Length Biased Quasi-Transmuted Uniform distribution is

$$\mu_k^1 = E(X^K) = \int_0^1 x^k \ g_{LBQTUD}(x)\, dx$$

$$= \frac{1}{a+3}\int_0^1 x^k[12ax^2 + 6(1-a)x]dx$$

$$= \frac{1}{a+3}[\frac{12ax^{k+3}}{k+3} + \frac{6(1-a)x^{k+2}}{k+2}]\Big|_0^1$$

$$= \frac{1}{a+3}\left[\frac{12a(k+2)+6(k+3)(1-a)}{(k+3)(k+2)}\right]$$

$$= \frac{1}{a+3}\left[\frac{6a(k+1)+6(k+3)}{(k+3)(k+2)}\right], a \in (0, \frac{1}{2}, 1)$$

Specifically,

$$\mu_1^1 = \mathrm{E}(X) = mean = \frac{1}{a+3}\left[\frac{a+2}{a+3}\right]$$

$$= \frac{2}{3}, \frac{5}{7} \& \frac{3}{4} \ for \ a \ = \ 0, \frac{1}{2} and \ 1 \ respectively$$

Also,

$$\mu_2^1 = \frac{1}{a+3}\left[\frac{9a+15}{10}\right]$$

$$= \frac{1}{2}, \frac{19.5}{35} \& \frac{6}{10} \ for \ a = 0, \frac{1}{2} \& 1 \ respectively$$

From the

$\mu_1^1$ *and* $\mu_2^1$ above, the variance of X, $\mathrm{V}(X) = E(X-\mu)^2 = \mu_2 = \mu_2^1 - (\mu_1^1)^2$ That is,

$\sigma_X^2 = 0.056, 0.047 \& 0.0375$ for a=0, $\frac{1}{2}$ &1 respectively .

The variance can be seen to reduce as the value of a is increasing. This is what is expected for a generalized distribution.

We note also that:

$$\mu_3^1 = \mathrm{E}(X^3) = \frac{1}{a+3}\left[\frac{4a+6}{5}\right]$$

$$= \frac{2}{5}, \frac{8}{17.5} \& \frac{1}{2} \textit{ for } a = 0, \frac{1}{2} \& 1 \textit{ respectively}$$

$$\mu_3 = \mathrm{E}(X-\mu)^3 = \mu_3^1 - 3\mu\mu_2^1 + 2\mu^3$$

$$= -0.007407, 0.007872 \textit{ and } -0.00625 \textit{ for } a = 0,$$

Combining $\mu_3^1$ *and* $\mu_3$ yield the coefficient of skewness, $\alpha_3 = \frac{\mu_3}{\sigma^3}$ .This implies that

$\alpha_3 = -0.5590, -0.77254$ and -0.86066 respectively for a =0, $\frac{1}{2}$ &1. The skewness can be seen to be reducing as the value of a is increasing. This is not the unexpected pattern for a generalized distribution.

Furthermore,

$$\mu_4^1 = \mathrm{E}(X^4) = \frac{1}{a+3}\left[\frac{5a+7}{7}\right]$$

$$= \frac{1}{3}, \frac{9.5}{24.5} \& \frac{3}{7} \textit{ for } a = 0, \frac{1}{2} \& 1 \textit{ respectively}$$

$$\mu_4 = \mathrm{E}(X-\mu)^4 = \mu_4^1 - 4\mu\mu_3^1 + 6\mu^2\mu_2^1 - 3\mu^4$$

$$= 0.007407, 0.006247 \textit{ and } 0.00435 \textit{ for } a = 0, \frac{1}{2} \& 1 \textit{ respectively}$$

Combining $\mu_4^1$ *and* $\mu_4$ yield the coefficient of kurtosis, $\alpha_4 = \frac{\mu_4}{\sigma^4}$ . This implies that

$\alpha_4 = 2.362, 2.828$ and 3.093 respectively for a =0, $\frac{1}{2}$ &1. The coefficient of kurtosis can be seen to be increasing toward 3.0 as the value of a increases from 0 to 1. The value of this coefficient became approximately 3.0 when a reached its maximum value of 1. This indicates that the length biased distribution being considered has the potentials of a generalized distribution.

### 2.1.2 Coefficients of Variation and Dispersion

The coefficients of variance and dispersion for the Length Biased Quasi-Transmuted distribution are $CV = \frac{\sigma}{E(X)}$ *and* $CD = \frac{\sigma^2}{E(X)}$. Specifically,

$$CV = \frac{\sqrt{\frac{-a^2+2a+5}{10(a+3)^2}}}{\left(\frac{a+2}{a+3}\right)} = 0.3536, 0.3033, 0.2582 \text{ for a=0}, \frac{1}{2} \,\&1$$

and $$CD = \frac{-a^2+2a+5}{10(a+3)(a+2)} = 0.083, 0.066, 0.05 \text{ for a=0}, \frac{1}{2} \,\&1$$

The two coefficients above can be seen to be reducing as the value of a is increasing. This is not the unexpected pattern for a generalized distribution.

### 2.1.3 Moment Generating Function

The moment generating function (m.g.f.) of a random variable X from the Length Biased Quasi-Transmuted Uniform distribution is defined as

$$M_X(t) = \mathrm{E}(e^{tX}) = \int_0^1 e^{tX} g_{LBQTUD}(x)dx$$

$$= \int_0^1 \left[1 + tX + \frac{(tX)^2}{2!} + \ldots\right] g_{LBQTUD}(x)dx$$

$$= \int_0^1 \sum_{j=0}^{\infty} \frac{t^j}{j!} X^j g_{LBQTUD}(x)dx$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j^1$$

$$= \frac{1}{a+3} \sum_{j=0}^{\infty} \frac{t^j}{j!} \left[\frac{12a(j+2)+6(j+3)(1-a)}{(j+3)(j+2)}\right], \text{ a } \varepsilon \ (0, \frac{1}{2}, 1)$$

### 2.1.4 Characteristic Function

The characteristic function (c. f.) of a random variable X from the Length Biased Quasi-Transmuted Uniform distribution is defined as

$$\phi_x(t) = M_X(it)$$
$$= \mathrm{E}(e^{itX})$$
$$= \int_0^1 e^{itX} g_{LBQTUD}(x)dx$$
$$= \frac{1}{a+3}\sum_{J=0}^{\infty}\frac{(it)^j}{J!}\left[\frac{12a(j+2)+6(j+2)(1-a)}{(j+3)(j+2)}\right], \text{ a } \varepsilon \ (0,\frac{1}{2},1)$$

## 2.1.5 Cumulant Generating Function

The cumulant generating function (c. g. f.) of a random variable X from the Length Biased Quasi-Transmuted Uniform distribution is defined as

$$K_X(t) = In\left[M_X(t)\right]$$
$$= In\left[\sum_{J=o}^{\infty}\frac{t^j}{j!}\mu_j^1\right]$$
$$= In\left[\frac{1}{a+3}\sum_{J=0}^{\infty}\frac{t^j}{j!}\left[\frac{12a(j+2)+6(j+2)(1-a)}{(j+3)(j+2)}\right]\right], \text{ a } \varepsilon \ (0,\frac{1}{2},1)$$

## 2.1.6 Hazard Function

The hazard function (h. f.) of a random variable X from the Length Biased Quasi-Transmuted Uniform distribution is defined as

$$h(x) = \frac{g_{LBQTUD}(x)}{1-G_{LBQTUD}(x)}$$
$$= \frac{12ax^2+6(1-a)x}{-4ax^3-3x^2(1-a)+a+3}, \text{a } \varepsilon \ (0,\frac{1}{2},1)$$

## 2.1.7 Reverse Hazard Function

The reserve hazard function (r. h. f.) of a random variable X from the Length Biased Quasi-Transmuted Uniform distribution is defined as

$$h_r(x) = \frac{g_{LBQTUD}(x)}{G_{LBQTUD}(x)}$$
$$= \frac{12ax^2+6(1-a)x}{4ax^3+3(1-a)x^2}, \ a \in (0,\frac{1}{2},1)$$

### 2.1.8 Survival Function

The survival function (s.f.) of a random variable X from the Length Biased Quasi-Transmuted Uniform distribution is defined as

$$S_X(x) = 1 - G_{LBQTUD}(x)$$

$$= \frac{-4ax^3 - 3x^2(1-a) + a + 3}{a+3}, a \in (0, \frac{1}{2}, 1)$$

### 2.1.9 Order Statistics

Suppose $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the order statistics from the random sample $X_1, X_2, \ldots, X_n$ from the Length Biased Quasi-Transmuted Uniform distribution with $g_{LBQTUD}(x)$ and $G_{LBQTUD}(x)$ as the pdf and cdf, the r$^{th}$ order statistic where $1 \le r \le n$ is given as

$$h_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!} g_{LBQTUD}(x)\left[G_{LBQTUD}(x)\right]^{r-1}\left[1 - G_{LBQTUD}(x)\right]^{n-r}$$

By setting r = n and r =1 in the function ($h_{(r)}(x)$) above, we have the distributions for the largest and lowest order statistics. For the largest order statistic we have that

$$h_{(r=n)}(x) = n\left[\frac{4ax^3 + 3(1-a)x^2}{a+3}\right]^{n-1}\left[\frac{12ax^2 + 6(1-a)x}{a+3}\right]$$

$$= \frac{n}{(a+3)^n}\left[4ax^3 + 3(1-a)x^2\right]^{n-1}\left[12ax^2 + 6(1-a)x\right], a \in (0, \frac{1}{2}, 1)$$

For

The lowest order statistic, we have that

$$h_{(r=1)}(x) = n\left[1 - G_{LBQTUD}(x)\right]^{n-1} g_{LBQTUD}(x)$$

$$= n\left[\frac{-4ax^3 - 3x^2(1-a) + a + 3}{a+3}\right]^{n-1}\left[\frac{12ax^2 + 6(1-a)x}{a+3}\right]^{n-1}$$

$$= \frac{n}{(a+3)^n}\left[-4ax^3 - 3x^2(1-a) + a + 3\right]^{n-1}\left[12ax^2 + 6(1-a)x\right], a \in (0, \frac{1}{2}, 1)$$

### 2.1.10 Random Number Generation

By the quantile function (i.e the inverse of the cdf), random numbers can be generated for the Length Biased Quasi-Transmuted Uniform distribution as follows.

$$Let\ \frac{1}{a+3}\left[4ax^3+3(1-a)x^2\right] = u \qquad so\ that$$

$$\frac{1}{a+3}\left[4ax^3+3(1-a)x^2\right]-u = 0 \Rightarrow 4ax^3+3(1-a)x^2-u(a+3)=0$$

*where U is a random variable from the Uniform* (0,1). *By setting* $a=0,\ 4ax^3+3(1-a)x^2-u(a+3)=0$ *becomes* $x^2=u \Rightarrow x=+\sqrt{u}$, *since* $U \sim Uniform(0,1)$. By setting a= $\frac{1}{2}$, $4ax^3+3(1-a)x^2-u(a+3)=0$ becomes $2x^3+1.5x^2$-3.5u=0. It should be noted that the roots of this equation changes with u. By setting a=1, $4ax^3+3(1-a)x^2-u(a+3)=0$ becomes $4x^3-4u=0 \Rightarrow x=+\sqrt[3]{u}$ since U~Uniform(0,1). The implication here is that for the random number generation, a must be set to 0 and 1.

### 2.1.11 Simulation Study

The table below gives the simulation results from the Length Biased Quasi-Transmuted Uniform distribution at a=0 and 1 for sample sizes 500, 1000, 2000 and 5000 respectively.

**Table 1.0a: Simulation from the LBQTUD**

| a =0 | | | a=1 | | |
|---|---|---|---|---|---|
| **Sample Size** | **Summary Statistics** | **Estimate** | **Sample Size** | **Summary Statistics** | **Estimate** |
| 500 | mean | 0.670 | 500 | mean | 0.764 |
| | variance | 0.055 | | variance | 0.033 |
| | standard deviation | 0.234 | | standard deviation | 0.181 |
| | skewness | -0.609 | | skewness | -0.842 |
| | kurtosis | 2.538 | | kurtosis | 3.071 |
| | coefficient of variation | 0.349 | | coefficient of variation | 0.237 |
| | coefficient of dispersion | 0.082 | | coefficient of dispersion | 0.043 |
| 1000 | mean | 0.672 | 1000 | mean | 0.757 |
| | variance | 0.056 | | variance | 0.037 |
| | standard deviation | 0.237 | | standard deviation | 0.191 |
| | skewness | -0.660 | | skewness | -0.909 |
| | kurtosis | 2.544 | | kurtosis | 3.256 |
| | coefficient of variation | 0.353 | | coefficient of variation | 0.252 |
| | coefficient of | | | coefficient of | |

| | | | | | |
|---|---|---|---|---|---|
| | dispersion | 0.083 | | dispersion | 0.049 |
| 2000 | mean | 0.667 | 2000 | mean | 0.745 |
| | variance | 0.056 | | variance | 0.036 |
| | standard deviation | 0.237 | | standard deviation | 0.190 |
| | skewness | -0.582 | | skewness | -0.774 |
| | kurtosis | 2.467 | | kurtosis | 2.972 |
| | coefficient of variation | 0.355 | | coefficient of variation | 0.256 |
| | coefficient of dispersion | 0.084 | | coefficient of dispersion | 0.049 |

**Table 1.0b: Simulation from the LBQTUD**

| **a=0** | | | **a=1** | | |
|---|---|---|---|---|---|
| **Sample Size** | **Summary Statistics** | **Estimate** | **Sample Size** | **Summary Statistics** | **Estimate** |
| 5000 | mean | 0.665 | 5000 | mean | 0.751 |
| | variance | 0.055 | | variance | 0.039 |
| | standard deviation | 0.235 | | standard deviation | 0.197 |
| | skewness | -0.550 | | skewness | -0.859 |
| | kurtosis | 2.379 | | kurtosis | 3.058 |
| | coefficient of variation | 0.353 | | coefficient of variation | 0.262 |
| | coefficient of dispersion | 0.083 | | coefficient of dispersion | 0.052 |

The simulation results in Tables (1.0)a and (1.0)b reveal concordance between the simulated results and the expected theoretical results; especially for the mean, variance, standard deviation, coefficient of variation and coefficient of dispersion. This further validates the postulated potentials of the Length Biased Quasi-Transmuted Uniform Distribution as a competent addition to the existing families of generalized distributions.

## 3.0 Conclusion

The specification and characterization of the Length Biased Quasi-Transmuted Uniform distribution has been successfully considered in this study. The variance, skewness and kurtosis were found to reduce as the value of a was increasing. Also, the coefficients of variation and dispersion decreased as the value of a increased as well. This is in line with the expected pattern observable for a generalized distribution when compared to the baseline distribution and some selected distributions belonging to the same family. The Length Biased Quasi-Transmuted Uniform Distribution (LBQTUD) is a therefore a worthy addition to the existing pool of generalized distributions.

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